Exterior powers of reflection representations

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Outline

1. Steinberg's theorem on reflection representations.
2. Our generalization.
3. Classification of reflection representations of Coxeter groups.
4. Apply the generalized Steinberg's theorem to Coxeter groups.

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Settings in Steinberg's theorem:

- V: an n-dim'l vector space with inner product $(-|-)$, e.g., a Euclidean space or a complex Hilbert space.
- $\{v_1, \ldots, v_n\}$: a basis of V (not necessarily orthonormal).
- W: the subgroup of $GL(V)$ generated by orthogonal reflections s_i w.r.t. v_i, $i = 1, \ldots, n$.

V is a W-module by the natural action.

The exterior power $\bigwedge^d V$, $d = 0, 1, \ldots, n$, admits a W-action

$$w(u_1 \wedge \cdots \wedge u_d) = (wu_1) \wedge \cdots \wedge (wu_d).$$

In particular, $\bigwedge^0 V$ is the 1-dim'l representation with trivial W-action, and $\bigwedge^n V$ is the 1-dim'l representation det.

Steinberg's theorem

Under these settings, we have:

Theorem (R. Steinberg, 1968)

Suppose V is a simple W-module. Then the W-modules

$$\bigwedge^d V, \quad d = 0, 1, \ldots, n,$$

are simple and pairwise non-isomorphic.

- The proof relies on the existence of the inner product which stays invariant under the W-action, and do induction on n.
  (Here $n = \dim V = \text{card}\,\{\text{defining generators of } W\}$.)
- A special case: (W, S) be an irreducible finite Coxeter group, and $V = V_{geom}$ be its geometric representation.
- The theorem of Steinberg can be generalized to the cases where the inner product does not exist.

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Definitions:

- A linear map s on an n-dim'l vector space V is called a *reflection* if:
  (1) $\exists$ a linear hyperplane H_s s.t. $s|_{H_s} = \mathrm{Id}_{H_s}$,
  (2) $\exists\ \alpha_s \in V \setminus \{0\}$ s.t. $s(\alpha_s) = \lambda_s \alpha_s$ for some $\lambda_s \neq 1$.
  α_s is called a *reflection vector* of s.
- Let W be a group and S be a set of generators. A representation $\rho : W \to GL(V)$ is called a *reflection representation* of (W, S) if:
  $\rho(s)$ is a reflection on V for any $s \in S$.

The first generalization

From now on, we work over a field F of char 0, and fix a group W with a finite set $S = \{s_1, \dots, s_m\}$ of generators.

Theorem (H. 2023)

Let (V, ρ) be an n-dim'l irreducible reflection representation of (W, S). Then the W-modules

$$\bigwedge^d V, \quad d = 0, 1, \dots, n,$$

are simple and pairwise non-isomorphic.

We require char $F = 0$ because we need the following:

Lemma (C. Chevalley, 1955)

Suppose W is a group and V, U are finite-dim'l semisimple W-modules over a field F of char 0. Then the W-module $V \bigotimes U$ is also semisimple.

Corollary

In our theorem, $\bigwedge^d V$ is semisimple since $\bigwedge^d V \hookrightarrow \bigotimes^d V$.

Instead of doing induction on n or m, our proof of simplicity is by showing

$$\mathrm{End}_W(\bigwedge^d V) = F$$

by linear algebra and some combinatorics on directed graphs.

The second generalization

Exterior powers of different reflection representations are also different.

Theorem (H. 2023)

Let (V_1, ρ_1) and (V_2, ρ_2) be two irreducible reflection representations of (W, S) of dim n_1 and n_2, resp. If

$$\bigwedge^{d_1} V_1 \simeq \bigwedge^{d_2} V_2$$

as W-modules for some d_1, d_2 with $0 < d_i < n_i$, then

$$d_1 = d_2 \text{ and } (V_1, \rho_1) \simeq (V_2, \rho_2).$$

Remarks:

- In most cases there is not a W-invariant inner product on V.
- The reflection vectors $\alpha_1, \ldots, \alpha_m$ are not necessarily a basis of V.

But the two points are crucial in Steinberg's proof of his theorem.

A Poincaré-like duality

Proposition (Who? Hu?)

Let $\rho : W \to GL(V)$ be an n-dim'l representation of W. Then

$$\bigwedge^{n-d} V \simeq (\bigwedge^{d} V)^* \bigotimes (\det \circ \rho)$$

as W-modules for all $d = 0, 1, \ldots, n$, where $(-)^*$ denotes the dual repn.

Question:

Is there any geometric interpretation (or in other manners) of this duality?

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Definition

Let S be a finite set.
Given $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for any $s, t \in S$ with $s \neq t$ such that $m_{st} = m_{ts}$.
The corresponding *Coxeter group* W is defined by a presentation

$$W = \langle s \in S \mid s^2 = e, \forall s \in S; \\ (st)^{m_{st}} = e, \forall s, t \in S \text{ with } m_{st} < \infty \rangle.$$

By convention, $m_{ss} := 1$, $\forall s \in S$.

• Historical interlude:

The concept of Coxeter groups originates from Euclidean reflection groups:

H. S. M. Coxeter proved in 1930's that any discrete reflection group in a Euclidean space admits such a presentation.

• Question:

Can we realize any Coxeter group as a reflection group on some space?

From now on, we fix a Coxeter group (W, S), and work over $\mathbb{R}$ or $\mathbb{C}$.

A refl. repn. of Coxeter groups (the geometric repn.)

Let $V_{geom} = \bigoplus_{s \in S} \mathbb{R}\alpha_s$ endowed with a bilinear form

$$(\alpha_s|\alpha_t) := -\cos\frac{\pi}{m_{st}}, \ \forall s, t \in S.$$

(Bourbaki 1968) V_{geom} is a reflection repn. of W via

$$s(\alpha_t) := \alpha_t - 2(\alpha_t|\alpha_s)\alpha_s.$$

The bilinear form $(-|-)$ is W-invariant, i.e.,

$$(wv|wu) = (v|u).$$

- Remark:

Although we have a W-invariant bilinear form on V_{geom}, but it is not an inner product in general. It is an inner product if and only if W is a finite group.

- Question:

Can we find out and classify all the reflection representations of (W, S)?

Recall that a reflection representation of a Coxeter group (W, S) is a representation $\rho : W \to GL(V)$ such that each $s \in S$ acts by a reflection.

For simplicity in presentation, we consider a specific class of reflection representations defined as follows.

Definition

If the reflection vectors $\{\alpha_s \mid s \in S\}$ form a basis for V, then we call (V, ρ) a *generalized geometric representation* of (W, S).

The classification of all refl. repn's of Coxeter groups

Theorem (informally presented, H. 2021)

The isomorphism classes of generalized geometric representations of (W, S) is parameterized by

$$\Big\{ \big((k_{st})_{s,t\in S, s\neq t}, \chi\big) \;\Big|\; k_{st}: \text{ certain "numbers"}, \\ \chi : H_1(\widetilde{G}, \mathbb{Z}) \to \mathbb{C}^\times \text{ a character} \Big\},$$

where $\widetilde{G}$ is certain simple graph determined by the numbers $(k_{st})_{s,t\in S, s\neq t}$,
and $H_1(\widetilde{G}, \mathbb{Z})$ is the first integral homology group of $\widetilde{G}$.

The classification of all refl. repn's of Coxeter groups

In general, any reflection representation of (W, S) can be described explicitly as well.

(It is a quotient representation of a generalized geometric representation of certain quotient Coxeter group of W, quotiented by a subrepresentation with trivial group action.)

The results below can also be stated for reflection representations in general.

Proposition (informally presented, H. 2021)

Let (V, ρ) be a generalized geometric representation of (W, S) corresponding to a datum

$$\big((k_{st})_{s,t\in S, s\neq t}, \chi\big)$$

where the numbers $(k_{st})_{s,t\in S, s\neq t}$ are "generic".
Then there exists a nonzero W-invariant bilinear form on V if and only if

$$\operatorname{Im}\chi \subseteq \{\pm 1\}.$$

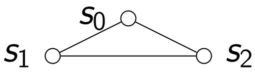
In other words, "most" reflection representation does not admit a nonzero W-invariant bilinear form.

Example: The affine Weyl group $\widetilde{A}_2$

- Let W be the affine Weyl group $\widetilde{A}_2$,

$$W = \langle s_0, s_1, s_2 \mid s_i^2 = (s_i s_j)^3 = e, \forall i, j. \rangle$$

k_{st}'s must be 1. The graph $\widetilde{G}$:



- The group $H_1(\widetilde{G}, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by the cycle.
- Characters of $H_1(\widetilde{G}, \mathbb{Z}) = \{\chi : \mathbb{Z} \to \mathbb{C}^\times\} = \mathbb{C}^\times$.
- Gen. geom. repn's $\xleftrightarrow{1:1} \mathbb{C}^\times$, and only two of them admit nonzero W-invariant bilinear form, i.e., the two corresponding to ± 1. (The standard geom. repn. $\leftrightarrow 1$)

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The homology group $H_1(\widetilde{G}, \mathbb{Z})$ is a finitely generated free abelian group.

Thus the set of characters $\chi : H_1(\widetilde{G}, \mathbb{Z}) \to \mathbb{C}^\times$ is identified to a torus

$$(\mathbb{C}^\times)^r \text{ where } r = \operatorname{rank} H_1(\widetilde{G}, \mathbb{Z}).$$

Under this identification, we have:

Proposition (informally presented, H. 2021)

Fix a set of "generic" numbers $(k_{st})_{s,t\in S, s\neq t}$ such that $\widetilde{G}$ is connected.
Then the characters of $H_1(\widetilde{G}, \mathbb{Z})$ corresponding to reducible generalized geometric representations form a "density zero" subset of $(\mathbb{C}^\times)^r$.

In other words, "most" reflection representations are irreducible.

(In the $\widetilde{A}_2$ example,
the generalized geometric representations $\xleftrightarrow{1:1}$ $\mathbb{C}^\times$,
only the one corresponding to 1 is reducible.)

Therefore, we can apply our generalization of Steinberg's theorem to all the irreducible reflection representations of (W, S).

Then we obtain much more irreducible representations of W which are pairwise non-isomorphic.

- Remark:

Some mild conditions on the reflection vectors $\{\alpha_s \mid s \in S\}$ yield that the reflection representations corresponds to Lusztig's $\boldsymbol{a}$-function value 1 (e.g., the generalized geometric representations.).

It would be interesting to consider the $\boldsymbol{a}$-function values of refl. repn's in general and their exterior powers.

Related papers

- H. Hu. On exterior powers of reflection representations.
  Bull. Aust. Math. Soc., online, 2023.
- H. Hu. On exterior powers of reflection representations, II.
  In preparation.
- H. Hu. Reflection representations of Coxeter groups and homology of Coxeter graphs.
  Algebr. Represent. Theory, accepted, arXiv:2306.12846, 2023.
- H. Hu. Representations of Coxeter groups of Lusztig's $\boldsymbol{a}$-function value 1.
  Preprint, arXiv:2309.00593, 2023.

Thank you for your attention!