Reflection representations of Coxeter groups

胡泓昇 / Hongsheng Hu

Beijing International Center for Mathematical Research, Peking University
(postdoc)

The Fourth International Conference on Groups, Graphs and Combinatorics
Southern University of Science and Technology, Shenzhen, China
Nov. 10–14, 2023

Outline

1. Backgrounds.
2. Classification of reflection representations of Coxeter groups.
3. Relation with Lusztig's function ***a*** (an informal section)
4. A generalization of Steinberg's theorem on reflection representations.

Outline

1. Backgrounds.
2. Classification of reflection representations of Coxeter groups.
3. Relation with Lusztig's function ***a*** (an informal section)
4. A generalization of Steinberg's theorem on reflection representations.

Let (W, S) be a Coxeter group of finite rank, that is:

- S is a finite set;
- W is a group defined by a presentation (i.e., generators and relations)

$$W = \langle s \in S \mid s^2 = e, \forall s \in S; \\ (st)^{m_{st}} = e, \forall s, t \in S \text{ with } m_{st} < \infty \rangle.$$

where $(m_{st})_{s,t \in S, s \neq t}$ are given elements in $\mathbb{N}_{\geq 2} \cup \{\infty\}$ such that $m_{st} = m_{ts}$.

By convention, $m_{ss} := 1$, $\forall s \in S$.

A Coxeter group (W, S) is uniquely determined by the Coxeter graph:

- set of vertices: S,
- set of edges: $s — t$ if $m_{st} \geq 3$, and the edge is labelled by m_{st}.

For example, the symmetric group $\mathfrak{S}_n$ with the generators $s_i := (i, i+1)$, $i = 1, \ldots, n-1$, is a Coxeter group with the corresponding Coxeter graph

with all edges are labelled by 3.

• Historical interlude:

The concept of Coxeter groups originates from Euclidean reflection groups. H. S. M. Coxeter proved in 1930's that any discrete reflection group in a Euclidean space admits such a presentation.

Q: Can we realize any Coxeter group as a reflection group on some space?

Let $V_{geom} = \bigoplus_{s \in S} \mathbb{R}\alpha_s$ endowed with a bilinear form

$$(\alpha_s | \alpha_t) := -\cos \frac{\pi}{m_{st}}, \quad \forall s, t \in S.$$

(c.f. Bourbaki 1968) V_{geom} is a reflection representation of W via

$$s(\alpha_t) := \alpha_t - 2(\alpha_t | \alpha_s)\alpha_s, \quad \forall s, t \in S.$$

The bilinear form $(-|-)$ is W-invariant, i.e., $(wv|wu) = (v|u)$.

- Remark:

Although we have a W-invariant bilinear form on V_{geom}, but it is not an inner product in general. It is an inner product if and only if W is a finite group.

Q: Can we find out and classify all the "reflection representations" of a Coxeter group (W, S)?

Outline

1. Backgrounds.
2. Classification of reflection representations of Coxeter groups.
3. Relation with Lusztig's function a (an informal section)
4. A generalization of Steinberg's theorem on reflection representations.

Definitions:

- An involutive linear map s on an finite dim'l vector space V is called a *reflection* if
  (1) there exists a linear hyperplane H_s such that $s|_{H_s} = \mathrm{Id}_{H_s}$,
  (2) there exists a nonzero vector α_s such that $s(\alpha_s) = -\alpha_s$.
- The hyperplane H_s is called the *reflection hyperplane* of s, and the vector α_s is called a *reflection vector* of s.
- A representation $\rho : W \to GL(V)$ is called a *reflection representation* of (W, S) if $\rho(s)$ is a reflection on V for any $s \in S$.

For simplicity in presentation, we assume $m_{st} < \infty$, $\forall s, t \in S$, and consider a specific class of reflection representations defined as follows.

Definition

Let (V, ρ) be a reflection representation of (W, S).
If the reflection vectors $\{\alpha_s \mid s \in S\}$ form a basis for V, then we call (V, ρ) a *generalized geometric representation* of (W, S).

In what follows we present the classification theorem over the base field $\mathbb{C}$. But the same results also hold over $\mathbb{R}$.

The classification of all refl. repn's of Coxeter groups

Theorem (H. 2021)

The isom. classes of gen. geom. repn's of (W, S) is parameterized by

$$\Big\{\big((k_{st})_{s,t\in S, s\neq t}, \chi\big) \;\Big|\; k_{st} = k_{ts} \in \mathbb{N}, 1 \leq k_{st} \leq \tfrac{m_{st}}{2},\ \forall s,t \in S, s \neq t; \\ \chi : H_1(\widetilde{G}, \mathbb{Z}) \to \mathbb{C}^\times \text{ is a character}\Big\},$$

where $\widetilde{G}$ is a simple graph determined by the numbers $(k_{st})_{s,t\in S, s\neq t}$:

- set of vertices: S,
- set of edges: $\{s — t \mid k_{st} < \frac{m_{st}}{2}\}$,

and $H_1(\widetilde{G}, \mathbb{Z})$ is the first integral homology group of $\widetilde{G}$.

In general, any reflection representation of (W, S) can be realized as a quotient representation (by a subrepresentation with trivial group action) of a generalized geometric representation of certain quotient group of W.

The homology group $H_1(\widetilde{G}, \mathbb{Z})$ is a finitely generated free abelian group.
Thus the set of characters $\chi : H_1(\widetilde{G}, \mathbb{Z}) \to \mathbb{C}^\times$ is identified to a torus

$$(\mathbb{C}^\times)^r \text{ where } r = \operatorname{rank} H_1(\widetilde{G}, \mathbb{Z}).$$

Under this identification, we have:

Proposition (H. 2021)

Fix a set of parameters $(k_{st})_{s,t \in S, s \neq t}$ such that the graph $\widetilde{G}$ is connected.

- The characters of $H_1(\widetilde{G}, \mathbb{Z})$ corresponding to reducible generalized geometric representations form a "density zero" subset of $(\mathbb{C}^\times)^r$. In other words, "most" gen. geom. repn's are irreducible.
- If a character χ corresponds to a reducible representation (V, ρ), then V has a maximal subrepresentation with trivial W-action, and the quotient is an irreducible reflection representation of (W, S).

There are only a few reflection representations admitting a nonzero W-invariant bilinear form.

Proposition (H. 2021)

Let (V, ρ) be a gen. geom. repn. of (W, S) corresponding to the datum

$$\big((k_{st})_{s,t\in S, s\neq t}, \chi\big).$$

Then there exists a nonzero W-invariant bilinear form on V if and only if

$$\operatorname{Im}\chi \subseteq \{\pm 1\}.$$

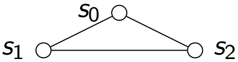
In other words, there is no nonzero W-invariant bilinear form on "most" generalized geometric representations.

Example: The affine Weyl group

- Let W be the affine Weyl group $\widetilde{\mathrm{A}}_2$,

$$W = \langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = (s_0 s_1)^3 = (s_1 s_2)^3 = (s_2 s_0)^3 = e. \rangle$$

Then the parameters k_{01}, k_{12}, k_{02} have no choices other than 1.

The graph $\widetilde{G}$: 

- The homology group $H_1(\widetilde{G}, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by the cycle in $\widetilde{G}$.
- The set of characters of $H_1(\widetilde{G}, \mathbb{Z}) = \{\chi : \mathbb{Z} \to \mathbb{C}^\times\} = \mathbb{C}^\times$.
- Gen. geom. repn's $\xleftrightarrow{1\,:\,1}$ $\mathbb{C}^\times$, and only two of them admit nonzero W-invariant bilinear form, i.e., the two corresponding to ± 1.

(The classical geom. repn. $\leftrightarrow$ 1, this is the only reducible gen. geom. repn.)

Remark:

In general, if $m_{st} = \infty$ for some $s, t \in S$, then in the classification of generalized geometric representations, the range $\mathbb{N} \cap [1, \frac{m_{st}}{2}]$ of the parameter k_{st} is replaced by

$$\mathbb{C} \cup \{*_1, *_2\},$$

and in the graph $\widetilde{G}$ we have an edge $s — t$ if and only if $k_{st} \neq 0$.

We also have similar results as previous ones.

Outline

1. Backgrounds.
2. Classification of reflection representations of Coxeter groups.
3. Relation with Lusztig's function $\boldsymbol{a}$ (an informal section)
4. A generalization of Steinberg's theorem on reflection representations.

• Coxeter group $(W, S) \rightsquigarrow$ Hecke algebra $\mathcal{H}$ (an alg. over $\mathbb{Z}[v^{\pm 1}]$)

After specialization $v \mapsto 1$, we obtain the group algebra $\mathcal{H} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{C} \simeq \mathbb{C}[W]$.

Thus a repn. (V, ρ) of W can be viewed as a repn. of $\mathcal{H}$ via $\rho(v) = 1$.

• $\mathcal{H}$ has a "good" basis $\{C_w \mid w \in W\}$ called Kazhdan–Lusztig basis, indexed by elements of W.

Structure constants of KL basis $\rightsquigarrow$ Lusztig's function $\boldsymbol{a} : W \to \mathbb{N}$

Definition

Let (V, ρ) be a representation of W. If there exists $n \in \mathbb{N}$ such that

- $\rho(C_w) = 0$ for any w with $\boldsymbol{a}(w) > n$,
- $\rho(C_w) \neq 0$ for some w with $\boldsymbol{a}(w) = n$,

then we say the representation (V, ρ) is of $\boldsymbol{a}$-function value n.

We give a characterization of representations of ***a***-function value 1.

Theorem (H. 2021)

A representation (V, ρ) of W is of ***a***-function value 1 $\iff \nexists\, v \in V \setminus \{0\}$ such that $s(v) = t(v) = -v$ for some $s, t \in S$ with $m_{st} < \infty$.

Corollary (H. 2021)

Let (V, ρ) be a reflection representation of (W, S) with reflection vectors $\{\alpha_s \mid s \in S\}$. Then (V, ρ) is of ***a***-function value 1 if and only if α_s and α_t are not proportional for any $s, t \in S$ with $m_{st} < \infty$.

We may guess that the ***a***-function value of a reflection representation is determined by the linearly independence relation of the reflection vectors.

For a specific class of Coxeter groups, we can determine all the irreducible representations of ***a***-function value 1.

Theorem (H. 2022)

Suppose (W, S) is simply laced and suppose there is at most one cycle on its Coxeter graph, then any irreducible representation of ***a***-function value 1 is "almost" a generalized geometric representation.

Here "almost" means that the reflection vectors $\{\alpha_s \mid s \in S\}$ span the representation space (not necessarily a basis) and they are not proportional to each other.

Such representations are proved to be a quotient of some gen. geom. repn. by a maximal subrepn. with trivial W-action, and they are classified in the same manner.

Outline

1. Backgrounds.
2. Classification of reflection representations of Coxeter groups.
3. Relation with Lusztig's function a (an informal section)
4. A generalization of Steinberg's theorem on reflection representations.

Notations:

- V: an n-dim'l vector space with inner product $(-|-)$,
  e.g., a Euclidean space or a complex Hilbert space.
- $\{v_1, \dots, v_n\}$: a basis of V (not necessarily orthonormal).
- W: the subgroup of $GL(V)$ generated by orthogonal reflections s_i
  w.r.t. v_i, $i = 1, \dots, n$.

V is a W-module by the natural action.

The exterior power $\bigwedge^d V$, $d = 0, 1, \dots, n$, admits a W-action

$$w(u_1 \wedge \cdots \wedge u_d) = (wu_1) \wedge \cdots \wedge (wu_d).$$

• Example:

(W, S): an irreducible finite Coxeter group,
$V = V_{geom}$: the geometric representation.

Theorem (R. Steinberg, 1968)

Suppose V is a simple W-module. Then the W-modules

$$\bigwedge^d V, \quad d = 0, 1, \ldots, n,$$

are simple and pairwise non-isomorphic.

The proof relies on the existence of the inner product which stays invariant under the W-action, and do induction on n.

The theorem of Steinberg can be generalized to the cases where the inner product does not exist.

The first generalization

Recall that a reflection representation of (W, S) is a representation (V, ρ) such that $\rho(s)$ is a reflection for each $s \in S$.

Theorem (H. 2023)

Let (V, ρ) be an n-dim'l irreducible reflection representation of (W, S). Then the W-modules

$$\bigwedge^d V, \quad d = 0, 1, \ldots, n,$$

are simple and pairwise non-isomorphic.

Unlike Steinberg's proof, our proof is done by the following two points:

(1) $\bigwedge^d V$ is semisimple,

(2) $\mathrm{End}_W(\bigwedge^d V) = \{\text{scalar multiplication}\}$,

and the point (2) uses some combinatorics on digraphs.

The second generalization

Exterior powers of different reflection representations are also different.

Theorem (H. 2023)

Let (V_1, ρ_1) and (V_2, ρ_2) be two irreducible reflection representations of (W, S) of dim n_1 and n_2, respectively. If

$$\bigwedge^{d_1} V_1 \simeq \bigwedge^{d_2} V_2$$

as W-modules for some d_1, d_2 with $0 < d_i < n_i$, then

$$d_1 = d_2 \text{ and } (V_1, \rho_1) \simeq (V_2, \rho_2).$$

Remarks:

(1) In most cases there is not a W-invariant inner product on V.

(2) The reflection vectors $\{\alpha_s \mid s \in S\}$ are not necessarily a basis of V.

Apply to Coxeter groups

Recall that "most" reflection representations of a Coxeter group are irreducible.

Therefore, we can apply our generalization of Steinberg's theorem to all the irreducible reflection representations of (W, S), and obtain a lot of pairwise non-isomorphic irreducible representations of W.

• It would be also an interesting problem to consider the $\boldsymbol{a}$-function values of these exterior powers.

An informal problem (from the rainbow Turán problem)

Consider a regular graph Γ of degree n, and suppose Γ admits a proper edge-coloring with n colors such that each color induces a perfect matching.

Then each color gives a permutation of vertices by swapping two vertices jointed by an edge of that color.

In this way the coloring gives an action of a Coxeter group on the vertices.

Let V be a vector space with a basis indexed by the set of vertices. Then V is a representation of the group which seems interesting to study.

Does this group-action help in considering the rainbow Turán problem of such a graph? e.g., avoiding rainbow cycles.

(This problem is communicated to me by Ruonan Li.)

Related papers

- H. Hu. Reflection representations of Coxeter groups and homology of Coxeter graphs. Preprint, arXiv:2306.12846, 2023.
- H. Hu. Representations of Coxeter groups of Lusztig's $\mathbf{a}$-function value 1. Preprint, arXiv:2309.00593, 2023.
- H. Hu. On exterior powers of reflection representations. Bull. Aust. Math. Soc., online, 2023.
- H. Hu. On exterior powers of reflection representations, II. In preparation.

Thank you for your attention!