Graphs and reflection representations

2023.09.14

AMSS seminar

I. Classifications and beyond.

Recall

H.S.M. Coxeter 1930's :

Any discrete reflection group (group generated by reflections) $W \subseteq O(E)$ on a Euclidean space E has a presentation of the form

$$W = \left\langle s_1, \cdots, s_n \;\middle|\; \begin{array}{ll} s_i^2 = id_E & \forall i \\ (s_i s_j)^{m_{ij}} = id_E & m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \quad \forall\, 1 \leq i < j \leq n \end{array} \right\rangle$$

where $s_i : E \to E$ is an orthogonal reflection w.r.t. some nonzero vector $\alpha_i \in E$

e.g. (finite, affine) Weyl groups of a s.s. Lie alg.

Rmk. Orthogonal refl:

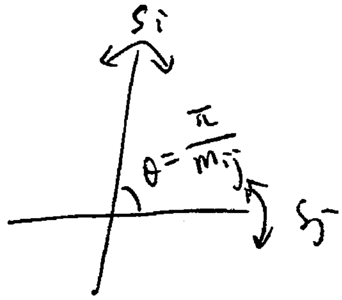
To emphasize that the refl. s_i is defined under the inner product on E

$$s_i : E \to E$$
$$x \mapsto x - \frac{2(x|\alpha_i)}{(\alpha_i|\alpha_i)} \alpha_i$$

The inner product stays invariant under the W-action. i.e.

$$(x|y) = (wx|wy) \quad \forall w \in W \quad x, y \in E.$$

Rmk the braid relation $(s_i s_j)^{m_{ij}} = id_E$



$s_i s_j$ = rotation by 2θ

- If θ is not a rational multiple of π, then $s_i s_j$ has infinite order

Inspired by Coxeter's result, we have the following defn

Def. Let S be a (finite) set.

Given $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for any pair of $s, t \in S$ such that $m_{st} = m_{ts}$,

a Coxeter group is defined to be a group with a presentation of the form

$$W = \langle s \in S \mid s^2 = e, \ \forall s \in S \quad (st)^{m_{st}} = e. \ \forall s, t \in S \ \text{ s.t. } m_{st} < \infty \rangle$$

So, this defn extends the notion of refl. gps on Euclidean spaces.

Q1. Can we realize an arbitrary Coxeter gp (W, S) as a "reflection group" on some vector space V?

(Is there a bilinear form on V inv. under the W-action s.t. the action of $s \in S$ is an "orthogonal refl"?)

We have an answer by J. Tits.

Let $V_{geom} := \bigoplus_{s \in S} \mathbb{R}\alpha_s$ be a vector space with formal basis $\{\alpha_s \mid s \in S\}$

We define a symmetric bilinear form $(-|-)$ on V_{geom} by:

$$\begin{cases} (\alpha_s \mid \alpha_s) = 1 & \forall s \in S \\ (\alpha_s \mid \alpha_t) = -\cos\frac{\pi}{m_{st}} & \forall s, t \in S, \ s \neq t \end{cases}$$

(Here we regard $-\cos\frac{\pi}{\infty} = -1$)

For any $s \in S$, define a linear map $\sigma_s \in GL(V_{geom})$ by $\sigma_s(x) = x - 2\frac{(x \mid \alpha_s)}{\cancel{}} \cdot \alpha_s$, $\forall x \in V_{geom}$

(In other words, σ_s is the orthogonal refl. w.r.t. α_s defined by the bilinear form $(-|-)$).

Then :

(Bourbaki, 1968) (V_{geom}, σ) is a well-defined representation of W, call the geometric rep, and (-1-) is inv. under the W-action.

Q2 Can we find out and classify all "such representations"?

(reps of W on which any $s \in S$ acts by an "abstract refl")

Def (1) Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$)

A linear map $s: V \to V$ is called a reflection if $V = H_s \oplus \mathbb{C}\alpha_s$ such that

$$s|_{H_s} = id_{H_s}, \quad s(\alpha_s) = -\alpha_s \neq 0.$$

(α_s: refl. vector, unique up to a scalar.

H_s: refl. hyperplane).

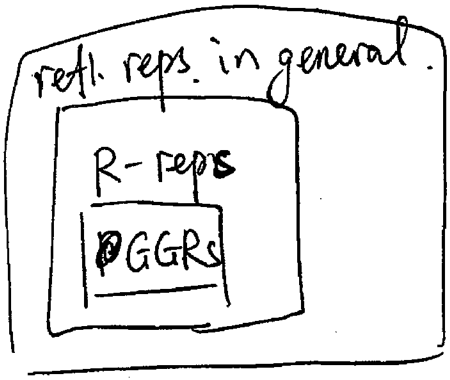
(2) Let V be a rep. of (W, S)

If $\forall s \in S$, s acts by a refl. on V. then V is called a <u>reflection representation</u> of W.

(3) If moreover the refl. ~~rep~~ vectors $\{\alpha_s \mid s \in S\}$ form a basis of V, then V is called a <u>generalized geometric representation</u> of W

(GGR for short)

<u>Rmk</u>. The "classification" of refl. reps is divided into three levels:



§ classification of GGRs.

For simplicity, we assume $m_{st} < \infty$. $\forall s, t \in S$

- Let $V = \bigoplus_{s \in S} \mathbb{C}\alpha_s$ be a GGR of (W, S)

Then
$$\begin{cases} s \cdot \alpha_t = \alpha_t + 2 b_{ts} \cos\frac{k_{st}\pi}{m_{st}} \alpha_s \\ t \cdot \alpha_s = \alpha_s + 2 b_{st} \cos\frac{k_{st}\pi}{m_{st}} \alpha_t \end{cases}$$

for some $k_{st} \in \mathbb{N}$, $1 \le k_{st} \le \frac{m_{st}}{2}$ ~~(unique)~~

$b_{ts}, b_{st} \in \mathbb{C}^\times$. $b_{st} \cdot b_{ts} = 1$ ~~(not unique)~~

- We define a graph $\tilde{G}$ as follows

vertex set : S

edge set : $s - t$ if $k_{st} \neq \frac{m_{st}}{2}$

(A subgraph of the Coxeter graph G)

- $\tilde{G}$ can be regarded as a simplicial complex of dim 1.

Thus we can consider $H_1(\tilde{G}, \mathbb{Z})$ which is a finitely generated free abelian group. The generators are cycles in the graph $\tilde{G}$.

- For a cycle c, say,

$$c = \begin{tikzcd} s_0 \arrow[dr, dash] \arrow[rrrr, dash] & & & & s_n \\ & s_1 \arrow[r, dash] & s_2 \arrow[r, dash] & \cdots \arrow[r, dash] & s_{n-1} \arrow[u, dash] \end{tikzcd}$$

in $\tilde{G}$, we define

$$\chi(c) := b_{s_0 s_1} b_{s_1 s_2} \cdots b_{s_{n-1} s_n} b_{s_n s_0} \in \mathbb{C}^\times .$$

This can be extended to a character

$$\chi : H_1(\tilde{G}, \mathbb{Z}) \to \mathbb{C}^\times$$

i.e. a group homomorphism.

Thm 1) The isom. classes of GGRs of (W, S) one-to-one correspond to the following set:

$$\left\{ \left((k_{st})_{s,t\in S,\, s\neq t},\ \chi \right) \;\middle|\; \begin{array}{l} k_{st} = k_{ts} \in \mathbb{N},\quad 1 \le k_{st} \le \frac{m_{st}}{2},\ \forall s,t \in S. \\ \chi : H_1(\tilde{G}, \mathbb{Z}) \to \mathbb{C}^\times \text{ is a character} \\ \text{where } \tilde{G} \text{ is the graph determined} \\ \text{by } (k_{st})_{s\neq t\in S} \end{array} \right\}$$

2) There exists a nonzero W-inv. bilinear form $(-|-)$ on a GGR V iff $\mathrm{Im}\,\chi \subseteq \{\pm 1\}$ where χ is the corr. character.

In this case, $(-|-)$ is symmetric and is unique up to a $\mathbb{C}^\times$-scalar, and

$$s(x) = x - 2\frac{(x|\alpha_s)}{(\alpha_s|\alpha_s)}\alpha_s$$

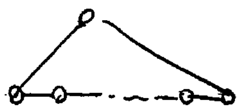
i.e. the action of s is an orthogonal reflection w.r.t. α_s defined by this bilinear form.

<u>Rmk</u> 1) If we drop the assumption $m_{st} < \infty$.

and if $m_{st} = \infty$, then in the classification the parameter set $\{1, 2, \dots, \frac{m_{st}}{2}\}$ is replaced by $\mathbb{C} \sqcup \underbrace{\{*_1, *_2\}}_{\rho_s^t,\ \rho_t^s}$

2) Only a few (finitely many) GGRs admit a W inv. bilinear form.

<u>E.g.</u> $\tilde{A}_n$



$H_1(\tilde{G}, \mathbb{Z}) \simeq \mathbb{Z}$

The GGRs of $\tilde{A}_n$ is parameterized by $\mathbb{C}^\times$

$$\left.\begin{array}{l} \text{The geometric rep} \leftrightarrow +1 \\ \boxed{} \leftrightarrow -1 \end{array}\right\} \text{admit } W\text{-inv. bilinear form.}$$

§ R-representations

Def. A refl. rep. V of W is called an R-representation if (1) V is spanned by refl. vectors $\{\alpha_s \mid s \in S\}$ as a vector space

(2) $\forall\ s, t \in S$, with $s \neq t$ and $m_{st} < \infty$, the vectors α_s and α_t are linearly independent.

Rmk. The defn. looks very weird at first glance. The original motivation comes from the investigation of Lusztig's function a:

$$a(R\text{-reps}) = 1$$

For certain simply laced Coxeter groups, it can be shown that any irrep of a-function value 1 must be an R-rep.

(Now we don't assume $m_{st} < \infty$)

<u>prop</u> Let $V = \sum_{s \in S} \mathbb{C}\alpha_s$ be an R-rep of W.

Then $\exists !$ GGR $\tilde{V}$ of W s.t.

$V \simeq \tilde{V} / V_0$ as W-reps

where $V_0 \subseteq \tilde{V}$ is a subrep. with trivial W-action

<u>Rmk</u> <u>1)</u>. It is not hard to compute the subspace $\tilde{V}_0$ of $\tilde{V}$ consisting of vectors fixed by W. In fact, it is the solution space of certain linear equations.

The space V_0 can be any subspace of $\tilde{V}_0$

<u>2)</u> If $m_{st} < \infty$. $\forall s, t \in S$. then the isom. classes of semisimple R-reps 1-1 corr. to the isom. classes of GGRs.

In particular, those simple R-reps corr. to those GGRs with connected graph $\tilde{G}$.

§ Refl. reps in general. (V, ρ)

Observation: If α_s and α_t are proportional for some $s, t \in S$ with $m_{st} < \infty$, then

$$\rho(s) = \rho(t)$$

<u>Prop</u> The isom. classes of refl. reps (spanned by refl. vectors) of W one-to-one corr. to the set

$$\left\{ \left(S = I_1 \sqcup \cdots \sqcup I_k, \ (V, \bar{\rho}) \right) \right\}$$

where <u>1)</u> $S = I_1 \sqcup \cdots \sqcup I_k$ is a partition of S such that

- $d_{ij} := \gcd \{ m_{rt} \mid r \in I_i, t \in I_j, m_{rt} < \infty \} > 1$ (by convention, $\gcd \phi = \infty$) for any $1 \le i \ne j \le k$
- $\forall i$, I_i can not be written as a disjoint union $I_i = J \sqcup J'$ s.t. $m_{rt} = \infty$ $\forall r \in J, \forall t \in J'$.

2) $\bar{\rho} : \bar{W} \to GL(V)$ is an $\mathbb{R}$-rep of $\bar{W}$, where $\bar{W}$ a Coxeter group of rank k defined by Coxeter metrix $(d_{ij})_{1 \le i, j \le k}$

The refl. rep of W corr. to $(I_1 \sqcup \cdots \sqcup I_k, (V, \bar{\rho}))$ is the composition

$$\rho : W \xrightarrow{\pi} \overline{W} \xrightarrow{\bar{\rho}} GL(V)$$

where π maps each $s \in I_i$ to the ith generator of $\overline{W}$.

<u>e.g.</u> If the partition of S is trivial (i.e., $k=1$. $S = I_1$)

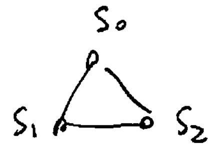
then the corr. refl. rep is the sign. rep.

<u>e.g.</u> The only admissible partition for A_3

are:
- the trivial partition
- the discrete ——
- $S = \{s_2\} \cup \{s_1, s_3\}$

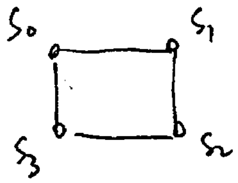
If $n \geq 4$. the only admissible partition for A_n are the trivial one and the discrete one.

e.g. $\tilde{A}_2$



$\{s_0\} \cup \{s_1, s_2\}$

$\tilde{A}_3$



$\{s_0, s_2\} \cup \{s_1, s_3\}$

$\tilde{A}_n \ (n \geq 4)$ only the trivial partition and the discrete —— .

§ images of refl. reps.

$\rho : W \hookrightarrow GL(V_{geom})$ the geom. rep.

$(-|-)$ on V_{geom} s.t.

$$(wx | wy) = (x|y) \quad \forall x, y \in V_{geom}, \ w \in W.$$

$$\Rightarrow \rho(W) \subseteq O(V_{geom}) = \{ \sigma \in GL(V_{geom}) \mid (\sigma - | \sigma -) = (-|-) \}$$

<u>Thm</u> (Y. Benoist, P. d. l. Harpe, 2004, Compositio)

Suppose (W, S) is irred. and of finite rank

If $(-|-)$ is non-positive and non-degenerate,

then the Zariski closure of $\rho(W)$ is $O(V_{geom})$

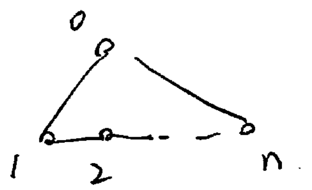
<u>Rmk</u> For GGRs, in general we don't have a

W-inv. bilinear form

So we can not talk about the "orthogonal gp".

Here are some examples.

<u>e.g.</u> $\tilde{A}_n$

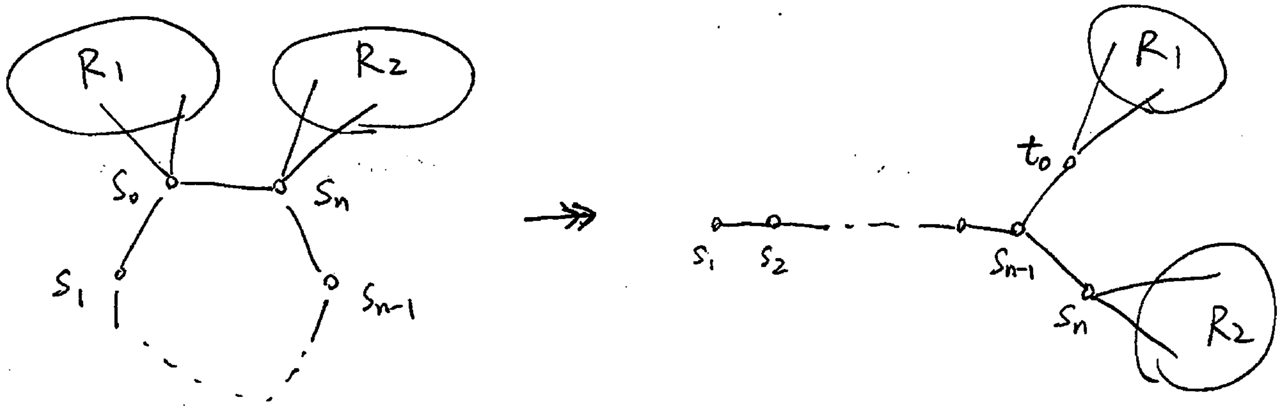


Let $\rho : \tilde{A}_n \to GL_{n+1}$ be the GGR corr. to -1.

Then the image $\rho(\tilde{A}_n) \simeq D_{n+1}$

(finite Weyl gp of type D)

By the same method, we have surjective homomorphisms of the form



Many finite and affine Weyl groups are of the form in the RHS. (n can be 2)

A_n. B_n $\tilde{B}_n$ $\tilde{C}_n$ $\tilde{D}_n$ D_n

E_n $\tilde{E}_n$. $\tilde{F}_4$. H_4.

<u>Q.</u> What can we say in general about the image (and its Zariski closure) of a GGR?

§ Inf. dim'l irreps.

<u>Thm</u> (An anonymous referee)

W: irred. Coxeter gp of finite rank

All irreps of W (over $\mathbb{C}$) are of finite dim iff W is a finite group or an affine Weyl gp.

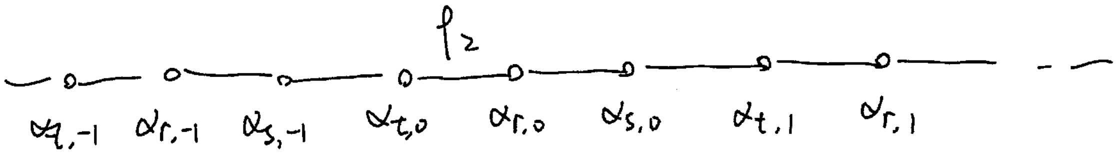
- The proof uses a result of Margulis and Vinberg saying that an inf. non-affine. irred Coxeter gp of finite rank admits a normal subgp Y of finite index such that Y has a quotient $Y \xrightarrow{\pi} F$ where F is a non-~~ab~~ abelian free group.
- Then we pull back an inf-dim'l irrep of F along π, and obtain an irrep. of Y.
  Then we induce it to W and find an irred component of inf. dim.
- But this only proves the existence.

For the following two kinds of Coxeter gps.
we can construct an inf-dim'l irrep explicitly

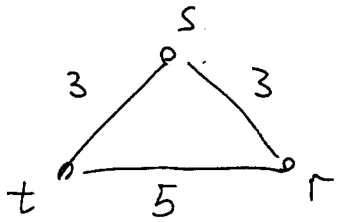
1) there are at least two cycles in the Coxeter graph
2) there is at least one cycle in the Coxeter graph
   and $m_{st} \geq 4$ for some $s, t \in S$.
   (No need for $s - t$ to be an edge on the cycle)

Illustrate the construction by two examples.

e.g.



$\downarrow$



$$V = \bigoplus_{n \in \mathbb{Z}} \left(\mathbb{C}\alpha_{t,n} \oplus \mathbb{C}\alpha_{r,n} \oplus \mathbb{C}\alpha_{s,n} \right)$$

$$r \cdot \alpha_{t,0} = \alpha_{t,0} + 2\cos\frac{2\pi}{5}\, \alpha_{r,0}$$

$$t\, \alpha_{r,0} = \alpha_{r,0} + 2\cos\frac{2\pi}{5}\, \alpha_{t,0}$$

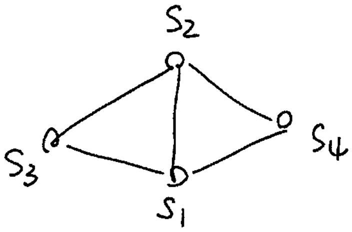
$$n \neq 0 \begin{cases} r \cdot \alpha_{t,n} = \alpha_{t,n} + 2\cos\frac{\pi}{5}\, \alpha_{r,n} \\ t\, \alpha_{r,n} = \alpha_{r,n} + 2\cos\frac{\pi}{5}\, \alpha_{t,n} \end{cases}$$

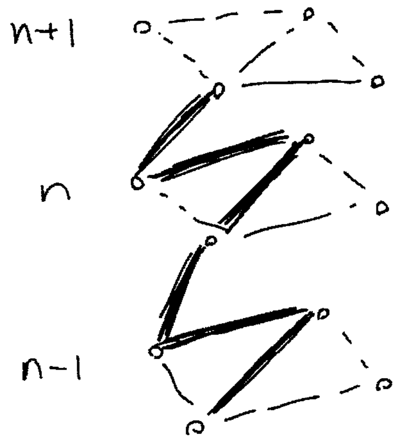
$$\forall n \begin{cases} r \cdot \alpha_{s,n} = \alpha_{s,n} + \alpha_{r,n} \\ s\, \alpha_{r,n} = \alpha_{r,n} + \alpha_{s,n} \end{cases} \qquad \forall n \begin{cases} t \cdot \alpha_{s,n} = \alpha_{s,n} + \alpha_{t,n+1} \\ s \cdot \alpha_{t,n} = \alpha_{t,n} + \alpha_{s,n-1} \end{cases}$$

then V/V_0 is an irrep of inf dim

where $V_0 = \{ v \in V \mid sv = rv = tv = v \}$ (=0 maybe)

e.g.





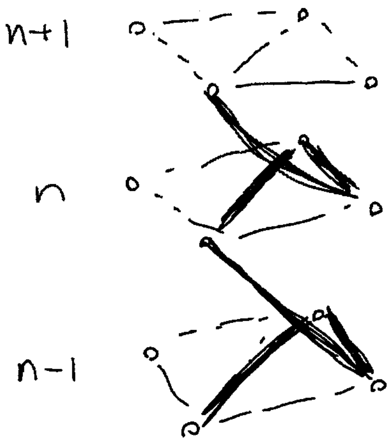
$$V := \bigoplus_{\substack{n \in \mathbb{Z} \\ i=1,\dots,4}} \mathbb{C}\alpha_{i,n}$$

$$\begin{cases} S_3 \alpha_{4,n} = \alpha_{4,n} \\ S_4 \alpha_{3,n} = \alpha_{3,n} \end{cases} \quad \forall n$$

$$\begin{cases} S_1 \alpha_{3,n} = \alpha_{3,n} + \alpha_{1,n+1} \\ S_3 \cdot \alpha_{1,n+1} = \alpha_{1,n+1} + \alpha_{3,n} \end{cases}$$

$$\begin{cases} S_1 \cdot \alpha_{4,n} = \alpha_{4,n} + z^n \alpha_{1,n+1} \\ S_4 \cdot \alpha_{1,n+1} = \alpha_{1,n+1} + z^{-n} \alpha_{4,n} \end{cases}$$

other $i \neq j$ $\begin{cases} S_i \alpha_{j,n} = \alpha_{j,n} + \alpha_{i,n} \\ S_j \alpha_{i,n} = \alpha_{i,n} + \alpha_{j,n} \end{cases}$



then V/V_0 is an irrep of inf. dim

where V_0 as above.

II Exterior powers

Thm (R. Steinberg. 1968)

Let E be a Euclidean space with inner product $(-|-)$.

and $e_1, \dots, e_n \in E$ be a set of basis

Let $s_1, \dots, s_n \in O(E)$ be orthogonal reflections w.r.t

$e_1, \dots, e_n$, resp. and $W = \langle s_1, \dots, s_n \rangle$ be the group

generated by $s_1, \dots, s_n$

Suppose E is an irrep of W

Then $\overset{d}{\wedge} E$, $0 \le d \le n$ are pairwise non-isomorphic

irreps of W

Rmk The proof relies on the inner product $(-|-)$

and is done by induction on the number of reflections n.

We will extend this result to a more general context.

where the W-inv. bilinear form may not exist.

__Sketched proof of Steinberg's thm__ :

Do induction on n

$n=1$ ✓

$n\geq 2$ Let G be a graph with n vertices $v_1, \cdots, v_n$

and $v_i - v_j$ be an edge iff e_i and e_j are not orthogonal

Then E is an irrep of $W \Leftrightarrow G$ is connected

$\exists$ a vertex (e.g., v_n) s.t. $v_1, \cdots, v_{n-1}$ span a connected subgraph in G

Let $U := \mathbb{R}\langle e_1, \cdots, e_{n-1}\rangle$ be the subspace of codim 1.

and $v \in E$ s.t. $v \perp U$

Then $E = U \oplus \langle v \rangle$ as a rep. of $W' := \langle s_1, \cdots, s_{n-1}\rangle$ $\boxed{\text{key point}}$

$(s_i \cdot v = v, \ \forall i = 1, \cdots, n-1)$

$$\bigwedge^d E = \left(\bigwedge^d U\right) \oplus \left[v \wedge \left(\bigwedge^{d-1} U\right)\right] \quad \text{as a rep of } W'$$

non-isomorphic irreps of W' by induction hypothesis

It is then not hard to show that one of them does not stay invariant under the action of s_n

$\Rightarrow \bigwedge^d E$ is an irrep of W. $(\forall\, 0 \leq d \leq n)$

A generalization:

<u>Thm</u> (Hu. 2023. Bull. Aust. Math. Soc.)

Let $\rho: W \to GL(V)$ be an n-dim rep of a group W over a field <u>of char. 0</u>

Suppose $s_1, \cdots, s_k \in W$ s.t.

(1) $\forall i$, $\rho(s_i)$ is a reflection

(2) $W = \langle s_1, \cdots, s_k \rangle$

(3) (V, ρ) is an irrep. of W

(4) $\forall i, j$ $\quad s_i \alpha_j \neq \alpha_j$ iff $s_j \alpha_i \neq \alpha_i$

Then $\{ \bigwedge^d V \mid 0 \leq d \leq n \}$ are pairwise non-isomorphic irreps of W.

Rmk • The condition (4) is a technical condition.

It looks like an orthogonality relation between α_i and α_j

But in general we don't have a W-inv. bilinear form on V

This would bring some trouble if we want to use the method of Steinberg.

• For example, suppose we have found a subspace $U \subseteq V$ of codim 1 spanned by reflection vectors $\alpha_1, \cdots, \alpha_{k-1}$, it may happen that the vector v which is fixed by $s_1, \cdots, s_{k-1}$ belongs to U.

and hence U is not an irrep of $\langle s_1, \cdots, s_{k-1} \rangle$,

and we don't have $V = U \oplus \langle v \rangle$

• The condition (4) is not that strict

If $s_i \alpha_j = \alpha_j$ and $s_j \alpha_i \neq \alpha_i$ then the order of $s_i s_j$ in W must be ∞

Moreover, there are uncountably many two-dim'l reps of $\langle s_i, s_j \rangle \simeq D_\infty$, but only two of them invalidate the condition (4).

• So our result applies to "most" reflection reps of Coxeter groups.

§ proof of our main thm.

- For a reflection s on V, we define

$$V_{d,s}^{+} := \{ v \in \bigwedge^{d} V \mid s.v = v \}$$

$$V_{d,s}^{-} := \{ v \in \bigwedge^{d} V \mid s.v = -v \}$$

- If $\bigwedge^{d} V \simeq \bigwedge^{d'} V$ as W-reps. then

$$\dim \bigwedge^{d} V = \dim \bigwedge^{d'} V \quad , \quad \dim V_{d,s}^{+} = \dim V_{d',s}^{+}$$

i.e. $\binom{n}{d} = \binom{n}{d'}$, $\binom{n-1}{d} = \binom{n-1}{d'}$

$\Rightarrow \quad d = d'$

$\therefore \{ \bigwedge^{d} V \mid 0 \le d \le n \}$ are pairwise non-isomorphic

- For irreducibility of $\bigwedge^{d} V$, we need:

Thm (Chevalley)

Let $\mathbb{F}$ be a field of char. 0. Let W be a group and V, U be fin-dim'l semisimple W-modules over $\mathbb{F}$.
Then $V \otimes U$ is also a semisimple W-module.

Note that $\bigwedge^d V \hookrightarrow \bigotimes^d V$ as a W-submod

<u>Cor</u> $\bigwedge^d V$ is semisimple

To show that $\bigwedge^d V$ is an irrep, it suffices to show

$$\mathrm{End}_W(\bigwedge^d V) \cong \mathbb{F}$$

- We define a graph $G=(S, E)$:

  vertex set : $S = \{1, \cdots, k\}$

  edge set : $E = \{ i - j \mid s_i \alpha_j \neq \alpha_j \}$

V irred $\Rightarrow \begin{cases} G \text{ is connected} \\ V = \langle \alpha_1, \cdots, \alpha_k \rangle \Rightarrow n = \dim V \leq k \end{cases}$

- <u>Claim</u> : $\exists\ I \subseteq S$ s.t.

  (1) $\{\alpha_i \mid i \in I\}$ is a basis of V

  (2) the subgraph $G(I)$ of G spanned by I is connected

It is proved inductively on the vertex set S.
In each step, we remove a vertex in S leaving the remaining subgraph connected, and the corresponding refl. vectors span the space V (This uses some techniques on graphs)

(But now V may not be an irrep of the subgp $\langle s_i \mid i \in I \rangle$)

- Now suppose $I = \{1, \dots, n\} \subseteq S$ is the subset obtained from the Claim

Then $\overset{d}{\wedge} V$ has a basis

$$\{ \alpha_{i_1} \wedge \dots \wedge \alpha_{i_d} \mid 1 \le i_1 < \dots < i_d \le n \}$$

<u>Claim</u>. For any indices $1 \le i_1 < \dots < i_d \le n$ the subspace $\bigcap_{1 \le j \le d} V_{d, s_{i_j}}^{-}$ of $\overset{d}{\wedge} V$ is one-dim'l with a basis vector $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_d}$

$$\left[\begin{aligned} &\text{for any } j = 1, \dots, d. \\ &s_{i_j} \cdot (\alpha_{i_1} \wedge \dots \wedge \alpha_{i_d}) = (\alpha_{i_1} + c_1 \alpha_{i_j}) \wedge \dots \wedge (-\alpha_{i_j}) \wedge \dots \wedge (\alpha_{i_d} + c_d \alpha_{i_j}) \\ &\qquad = \alpha_{i_1} \wedge \dots \wedge (-\alpha_{i_j}) \wedge \dots \wedge \alpha_{i_d}. \end{aligned} \right.$$

- Therefore, for any $\varphi : \overset{d}{\wedge} V \to \overset{d}{\wedge} V \in \mathrm{End}_W(\overset{d}{\wedge} V)$

$$\varphi(\alpha_{i_1} \wedge \dots \wedge \alpha_{i_d}) = \gamma_{i_1, \dots, i_d} \, \alpha_{i_1} \wedge \dots \wedge \alpha_{i_d} \text{ for some } \gamma_{i_1, \dots, i_d} \in \mathbb{F}$$

So it suffices to show that

$\gamma_{i_1, \dots, i_d}$ is independent of the indices $1 \le i_1 < \dots < i_d \le n$

i.e for any two subsets $I_1 = \{1 \le i_1 < \dots < i_d \le n\}$ $I_2 = \{1 \le j_1 < \dots < j_d \le n\}$ of $I = \{1, \dots, n\}$,

$$\gamma_{i_1, \dots, i_d} = \gamma_{j_1, \dots, j_d}$$

• Recall that $I = \{1, \dots, n\} \subseteq S = \{1, \dots, k\}$

and the subgraph $G(I)$ is connected

<u>Def</u> We say I_2 is obtained from I_1 by a move in $G(I)$

if $\exists\, i \in I_1,\ j \in I_2$ s.t. $I_1 \setminus \{i\} = I_2 \setminus \{j\}$

and $i - j$ is an edge in $G(I)$

<u>Claim</u> 1) If I_2 is obtained from I_1 by a move in $G(I)$

then $\gamma_{i_1, \dots, i_d} = \gamma_{j_1, \dots, j_d}$

2) Let $I_1, I_2 \subseteq I$ be any two subsets

s.t. $|I_1| = |I_2| = d$

Then I_2 can be obtained from I_2 by finite

steps of moves in $G(I)$

1) is proved by computing

$$s_j \cdot \varphi(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}) = \varphi\big(s_j(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d})\big)$$

and comparing the coefficients in the two sides

2) is proved combinatorically on the $\underbrace{\text{graph}}_{\text{connected}}$ $G(I)$.

$\square$

§ other results

- $s_1, \dots, s_k : V \to V$ reflections

  then $\displaystyle\bigcap_{1\le i\le k} V^+_{d,s_i} = \bigwedge^d \Big(\bigcap_{1\le i\le k} H_i\Big)$, $\forall\, 0 \le d \le n$

  where $H_i \subseteq V$ the refl. hyperplane of s_i

- a Poincare-like duality:

  $\rho : W \to GL(V)$ an n-dim'l rep.

  then $\bigwedge^{n-d} V \simeq \big(\bigwedge^d V\big)^* \otimes (\det \circ \rho)$ as W-reps

  $\forall\, 0 \le d \le n$.

§ Further questions

1) Can we remove the technical condition (4) in the main thm?

(4): $\forall i, j.\quad s_i\alpha_j \neq \alpha_j \iff s_j\alpha_i \neq \alpha_i$

2) Is it possible to find two non-isom refl. reps V_1 and V_2 of W satisfying the conditions of the main thm and two integers d_1 d_2 with $0 < d_i < \dim V_i$ such that $\bigwedge^{d_1} V_1 \simeq \bigwedge^{d_2} V_2$ as W-reps?

If NOT, then we can obtain so many irreps of a Coxeter groups.

If YES, what can we say on this isom?

3) What kinds of group reps V have the property that $\bigwedge^d V$, $0 \leq d \leq \dim V$, are pairwise non-isom irreps?

Any necessary conditions?