Asymptotic log-concavity of dominant lower Bruhat intervals

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The shape of Bruhat intervals

Björner and Ekedahl pioneered the study of length-counting sequences associated with (parabolic) lower Bruhat intervals in crystallographic Coxeter groups [1]. In this work, we study the asymptotic behavior of these sequences in affine Weyl groups. Let $W = \mathbb{Z}\Phi^\vee \rtimes W_f$ be the affine Weyl group with Weyl group W_f and root system Φ of rank r. Let ${}^fW$ be set of minimal representatives for the right cosets $W_f \backslash W$. Let C_+ be the dominant Weyl chamber. Let t_λ be the translation by a dominant coroot lattice element $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ and ${}^fb_i^{t_\lambda}$ be the number of elements of length i below t_λ in the Bruhat order on ${}^fW$, which is the $2i$-dimensional Betti number of a (spherical) Schubert variety in the affine Grassmannian $\mathcal{G}r := G(F)/G(\mathcal{O})$, where $F = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let $K = G(\mathcal{O})$. We have the Bruhat decomposition

$$\mathcal{G}r = \bigsqcup_{\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}} \mathcal{G}r_\lambda \text{ where } \mathcal{G}r_\lambda := Kt^\lambda K/K.$$

We regard t^λ as a point in $G(\mathbb{C}[t^{\pm 1}]) \subset G(F)$. The corresponding **spherical Schubert variety** is

$$\overline{\mathcal{G}r_\lambda} = \bigsqcup_{\mu \in \mathbb{Z}\Phi^\vee \cap \overline{C_+},\ \mu \preceq \lambda} \mathcal{G}r_\mu,$$

where $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a sum of positive coroots.

Key pieces: P^λ, $\mathfrak{m}_k$, and S_k

Let $\lambda \in \mathbb{Z}\Phi^\vee \cap \overline{C_+}$ be a fixed dominant coroot lattice element. We define the convex polytope

$$P^\lambda := \mathrm{Conv}\{w\lambda \mid w \in W_f\} \cap \overline{C_+} \subset E,$$

where Conv denotes the convex hull of a set. Let $\mathrm{ht}\colon P^\lambda \to \mathbb{R}$ be the height function $\mathrm{ht}(x) := (2\rho|x)$, where ρ is the half-sum of positive roots. We denote by Vol_r the Lebesgue measure on E and by $\mathrm{ht}_*\mathrm{Vol}_r$ the corresponding push-forward measure on $\mathbb{R}$. Then, the density function of $\mathrm{ht}_*\mathrm{Vol}_r$, which is

$$g(z) = \frac{1}{\|2\rho\|}\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(z)), \quad z \in \mathbb{R},$$

evaluates volumes of hyperplane sections of the polytope P^λ up to a scalar. Let δ_z denote the Dirac measure (that is, point mass) at the point $z \in \mathbb{R}$. For any positive integer k, we define the discrete measure $\mathfrak{m}_k$ supported on $[0, \ell(t_\lambda)]$ by

$$\mathfrak{m}_k := \frac{1}{k^r} \sum_{0 \le i \le k\ell(t_\lambda)} {}^fb_i^{t_{k\lambda}} \delta_{\frac{i}{k}}.$$

Intuitively, $\mathfrak{m}_k$ distributes the sequence $({}^fb_i^{t_{k\lambda}})_i$ evenly on the interval $[0, \ell(t_\lambda)]$. We also define the step function $S_k\colon [0, \ell(t_\lambda)] \to \mathbb{R}$ by

$$S_k(z) := \frac{1}{k^{r-1}} {}^fb_i^{t_{k\lambda}}, \text{ whenever } z \in \left[\frac{i}{k}, \frac{i+1}{k}\right).$$

The function S_k records the numbers $({}^fb_i^{t_{k\lambda}})_i$ and behaves like the "density function" of $\mathfrak{m}_k$.

Our Results

Let $\mathrm{Vol}_r(A_+)$ be the volume of the fundamental alcove A_+.

Main Theorem

The weak convergence of $(\mathfrak{m}_1, \mathfrak{m}_2, \ldots)$. The sequence of measures $(\mathfrak{m}_k)_k$, as k tends to infinity, converges weakly to the measure $\frac{1}{\mathrm{Vol}_r(A_+)}\mathrm{ht}_*\mathrm{Vol}_r$.
The uniform convergence of $(S_1, S_2, \ldots)$. The sequence of functions $(S_k)_k$, as k tends to infinity, converges uniformly to $\frac{1}{\mathrm{Vol}_r(A_+)}g$.

Let $\pi^{t_\lambda}(q)$ be the generating polynomial of the sequence $({}^fb_i^{t_\lambda})_i$, which is the Poincaré polynomial of the singular cohomology of the spherical Schubert variety $\overline{\mathcal{G}r_\lambda}$ in the affine Grassmannian. Let ${}^\mu W_f$ be the set of minimal representatives for the right cosets $W_{f,\mu}\backslash W_f$, and $W_{f,\mu}$ is the stabilizer of μ in W_f.

The dominant lattice formula

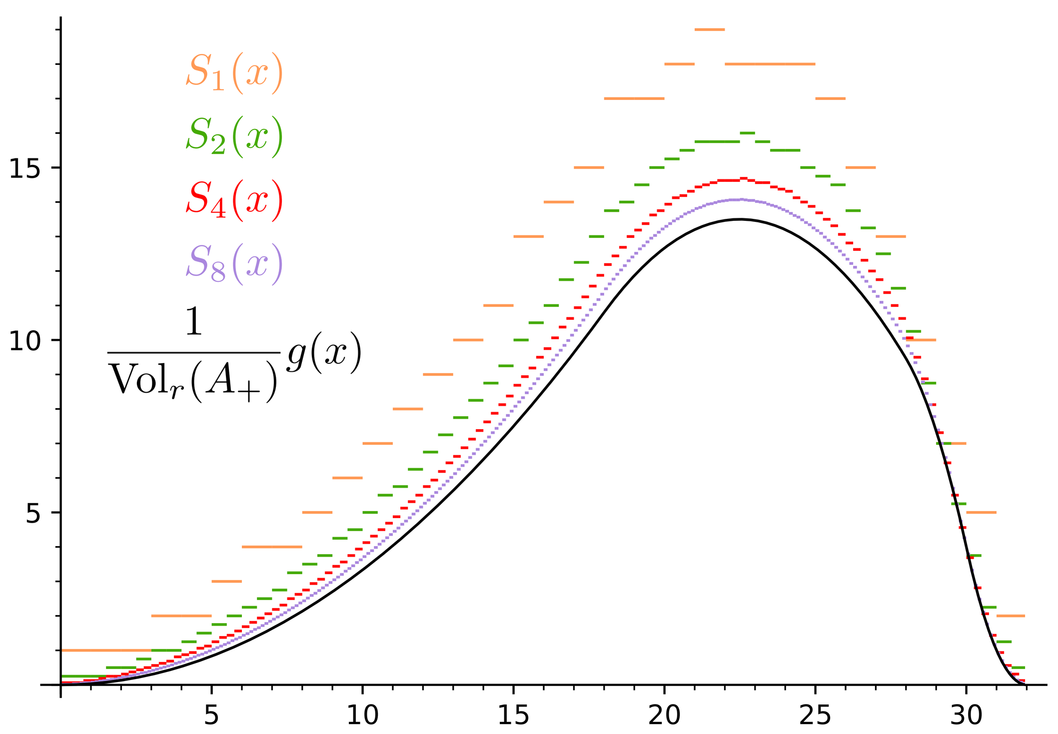
$$\pi^{t_\lambda}(q) = \sum_{\mu \in P^\lambda \cap \mathbb{Z}\Phi^\vee} q^{(2\rho|\mu)} \cdot \sum_{w \in {}^\mu W_f} q^{-\ell(w)}$$

By the Brunn–Minkowski inequality, we obtain:

Asymptotic log-concavity

The density function g is log-concave. That is, the sequence $({}^fb_i^{t_{k\lambda}})_i$ is asymptotically log-concave as k tends to infinity.

An example in type C_3



Let $\lambda = 3\alpha_1^\vee + 6\alpha_2^\vee + 7\alpha_3^\vee$ so $\mathrm{ht}(\lambda) = 32$. For convenience, we define $(a,b,c)_\Phi := a\alpha_1^\vee + b\alpha_2^\vee + c\alpha_3^\vee$. The polytope P^λ has vertices

$$\{(0,0,0)_\Phi, (3,3,3)_\Phi, (3,5,7)_\Phi, (3,6,6)_\Phi, (7/3, 14/3, 7)_\Phi, (3,6,7)_\Phi\},$$

which is an example of a non-lattice polytope. Since $\rho = (3,5,3)_\Phi$, we get $\|\rho\| = \sqrt{14}$. By direct computations, we have $\mathrm{Vol}_3(A_+) = 1/48$. The only missing ingredient to compute the limit function is to determine $\mathrm{Vol}_2(\mathrm{ht}^{-1}(z))$. By the theory of convex polytopes, this function is a piece-wise quadratic polynomial.
We can use the "uniform convergence" to give quick estimates of the terms in the sequence $({}^fb_i^{t_{k\lambda}})_i$ when k is big. For instance, when $k = 8$, the value of ${}^fb_{196}^{t_{8\lambda}}$ is virtually impossible to obtain in a computer directly from definitions. Let us take $z = 24.5 (= 196/8)$. By our theorem, we have

$$S_8(24.5) = \frac{1}{8^2}{}^fb_{196}^{t_{8\lambda}} \sim 48g(24.5) = \frac{389}{30},$$

which gives ${}^fb_{196}^{t_{8\lambda}} \sim 829.87$.
On the other hand, the "dominant lattice formula" gives the exact values of the terms in the sequence $({}^fb_i^{t_{k\lambda}})_i$. In particular, we have ${}^fb_{196}^{t_{8\lambda}} = 863$. Our quick estimate of 829.87 was off by 3.84%.

Connection with Ehrhart's theory

For an r-dimensional lattice polytope P (that is, all vertices of P are points of a given lattice L), the **Ehrhart polynomial** [2] $E(P,k)$ is a polynomial in k that counts the number of lattice points in the k-fold dilation kP of P.

The leading coefficient is equal to the r-dimensional volume $\mathrm{Vol}_r(P)$ of P, divided by the volume $d(L)$ of the fundamental region of the lattice L.

Question

For k sufficiently large, is the total Betti number

$$\mathrm{Card}\left({}^f[e, t_{k\lambda}]\right) = \sum_i {}^fb_i^{t_{k\lambda}}$$

a quasi-polynomial in k of degree r, with $\frac{\mathrm{Vol}_r(P^\lambda)}{\mathrm{Vol}_r(A_+)}$ as the leading coefficient?

Question

Is ${}^fb_{ki}^{t_{k\lambda}}$ a quasi-polynomial in k of degree $(r-1)$ for k sufficiently large, with

$$\frac{\mathrm{Vol}_{r-1}(\mathrm{ht}^{-1}(i))}{\mathrm{Vol}_r(A_+) \cdot \|2\rho\|}$$

as the leading coefficient?

References

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