Coxeter Groups and Their Reflection Representations

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Beijing International Center for Mathematical Research

Postdoc Seminar
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Outline

- The concept of Coxeter groups is an algebraic abstract of reflection groups.
- Reflection representations of Coxeter groups.
- Relationship to Lusztig's a-function; infinite dimensional representations.

Reflections

- $E = \mathbb{R}^n$ Euclidean space, with inner product $(\cdot|\cdot)$.
- $\alpha \in E$ such that $|\alpha| = 1$.
- $H_\alpha = \langle\alpha\rangle^\perp \ni 0$ the hyperplane perpendicular to α.
- The (linear) reflection w.r.t. α is a linear map σ_α defined by

$$\begin{aligned}\sigma_\alpha : E &\to E \\ x &\mapsto x - (x|\alpha)\alpha\end{aligned}$$

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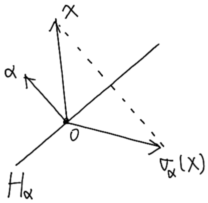

Reflections

Facts

1. $\sigma_\alpha(\alpha) = -\alpha$.
2. $\sigma_\alpha|_{H_\alpha} = \mathrm{Id}_{H_\alpha}$.
3. $\sigma_\alpha^2 = \mathrm{Id}_E$, thus $\sigma_\alpha \in \mathrm{GL}(E)$.
4. $(\sigma_\alpha x|\sigma_\alpha y) = (x|y)$, $\forall x, y \in E$, i.e. $\sigma_\alpha \in \mathrm{O}(E)$.
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Question: Given a set of reflections $S = \{\sigma_{\alpha_i} | i \in I\}$, what is the group $\langle S \rangle$ generated by S?

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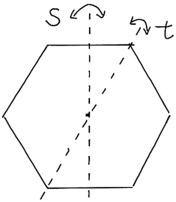
Example: Symmetric group of a regular hexagon (六边形)



- s, t two reflections $\Longrightarrow$ $s^2 = t^2 = \mathrm{Id}$.
- st = rotation by angle $\pi/3$ (anticlockwise) $\Longrightarrow$ $(st)^6 = \mathrm{Id}$, i.e. $ststst = tststs$ (braid relation).
- The symmetric group $\mathbb{S}$ of the hexagon consists of

$$\{\mathrm{Id}, st, (st)^2, (st)^3, (st)^4, (st)^5, s, (st)s, (st)^2 s, (st)^3 s, (st)^4 s, (st)^5 s\}$$

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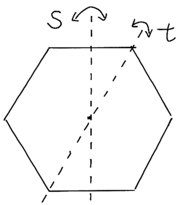

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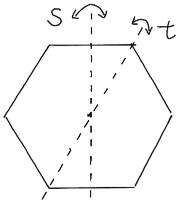

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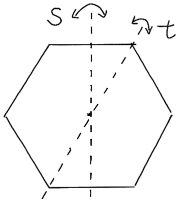

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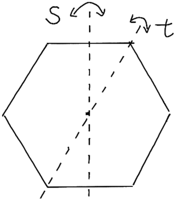

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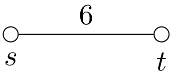
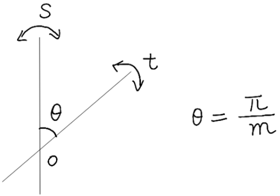


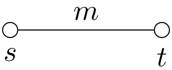
Example: Finite dihedral (二面体) groups



More generally, a finite dihedral group $\mathrm{I}_2(m)$ is defined by

$$\mathrm{I}_2(m) = \langle s, t | e = s^2 = t^2 = (st)^m \rangle$$

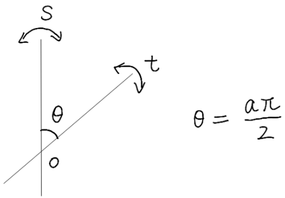
The corresponding graph:



Example: Infinite dihedral group

Suppose in the Euclidean plane, we have two lines forming an angle $a\pi/2$, where $a \in (0,1) \setminus \mathbb{Q}$,

then $st =$ rotation by $a\pi \implies \nexists m \in \mathbb{N}$ s.t. $(st)^m = \mathrm{Id}$.



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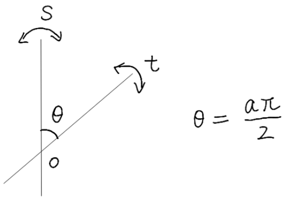

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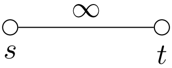
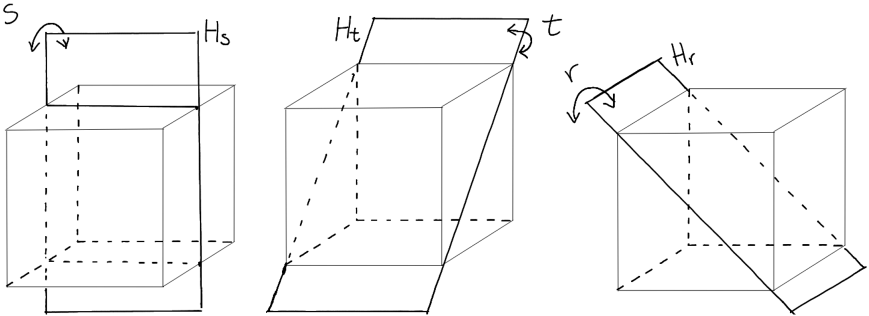


Example: Symmetric group of a cube



- s, t, r three reflections $\implies s^2 = t^2 = r^2 = \mathrm{Id}.$

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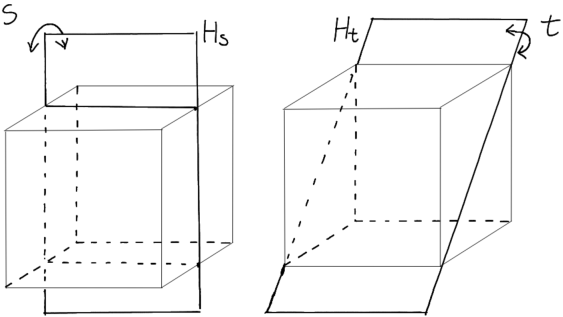
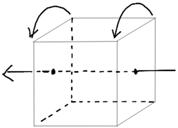


- st = a rotation by $\pi/2$: $\Longrightarrow$ $(st)^4 = \mathrm{Id}.$



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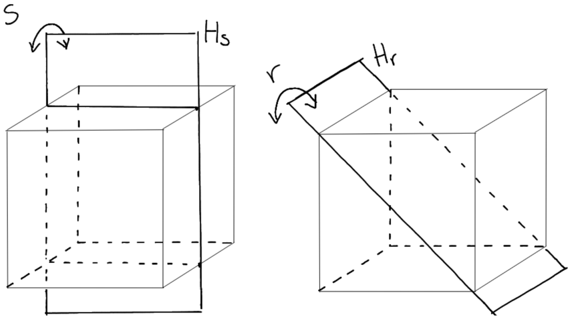
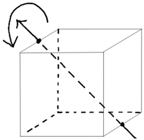


- $sr =$ a rotation by π: $\Longrightarrow$ $(sr)^2 = \mathrm{Id}$, i.e. $sr = rs$.



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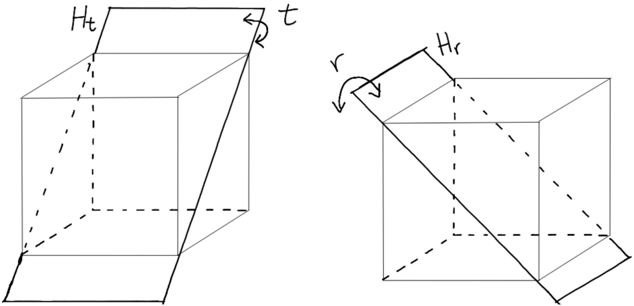
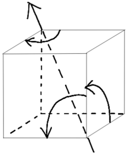


- tr = a rotation by $2\pi/3$: $\implies (tr)^3 = \mathrm{Id}.$



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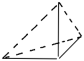
- The symmetric group $\mathbb{S}$ of the cube is generated by s, t, r. (Since any automorphism of the cube is determined by the image of the corner , and one can transfer this corner to anywhere in any posture using compositions of s, t, r.)
- In fact,

$$\mathbb{S} = \langle s, t, r | e = s^2 = t^2 = r^2 = (st)^4 = (sr)^2 = (tr)^3 \rangle$$

(It would take some effort to see this. One could write down all elements; alternatively, use the theory of root system.)

- The corresponding graph:

There is no edge between s and r, indicating $sr = rs$ commuting.

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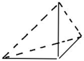
- The symmetric group $\mathbb{S}$ of the cube is generated by s, t, r. (Since any automorphism of the cube is determined by the image of the corner , and one can transfer this corner to anywhere in any posture using compositions of s, t, r.)
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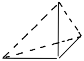
- The symmetric group $\mathbb{S}$ of the cube is generated by s, t, r. (Since any automorphism of the cube is determined by the image of the corner , and one can transfer this corner to anywhere in any posture using compositions of s, t, r.)
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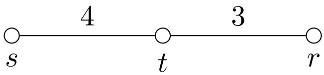

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Example: Symmetric group $\mathfrak{S}_n$ of $\{1, 2, \ldots, n\}$

Let $s_i = (i, i+1)$ be the transposition, $i = 1, 2, \ldots, n-1$.

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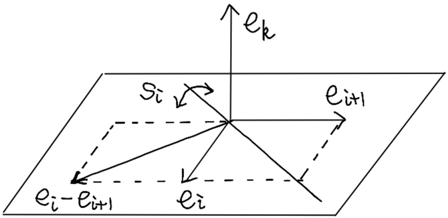
$s_1, \ldots, s_{n-1}$ can be realized as Euclidean reflections as well.

$$\mathfrak{S}_n \hookrightarrow \mathrm{GL}(\mathbb{R}^n)$$
$$s_i \mapsto \text{swapping } e_i \text{ and } e_{i+1}$$
$$\text{i.e. reflection w.r.t. } e_i - e_{i+1}$$

where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.



Coxeter groups

Each presentation we have seen is determined by a set S of *involutions* and a set of numbers $\{m_{st}\}_{s,t\in S, s\neq t}$ such that $\forall s,t \in S$, $s \neq t$,

- $m_{st} = m_{ts}$,
- $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

Definition

Given S and (m_{st}) as above, the group

$$W = \langle S | s^2 = e, \forall s \in S; (st)^{m_{st}} = e, \forall s,t \in S, s \neq t \rangle$$

is called a Coxeter group.

Here the relation $(st)^{m_{st}} = e$ is equivalent to $\underbrace{sts\cdots}_{m_{st}\text{ factors}} = \underbrace{tst\cdots}_{m_{st}\text{ factors}}$ (braid relation).

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Coxeter graphs

The presentation can be encoded in a labelled graph:

- Vertices set: S.
- Labelled edges: $s \overset{m_{st}}{\text{———}} t$ if $m_{st} \geq 3$.
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This graph is called the Coxeter graph of (W, S).

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1. Let E be a Euclidean space. Any *discrete* subgroup of $\mathrm{GL}(E)$ generated by (linear) reflections is finite, and has a presentation of the form

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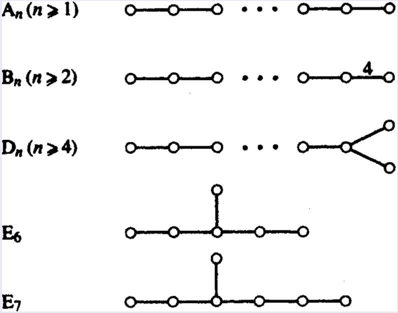

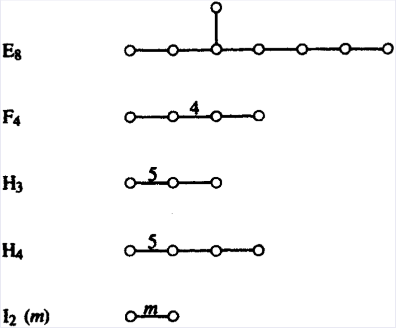

Platonic solids

There are only 3 finite Coxeter groups with connected Coxeter graph such that $|S| = 3$, i.e. A_3, B_3, H_3.

The group of type A_3 is $\mathfrak{S}_4$, which is the symmetric group of a regular tetrahedron (四面体).

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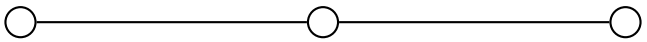
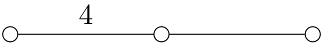
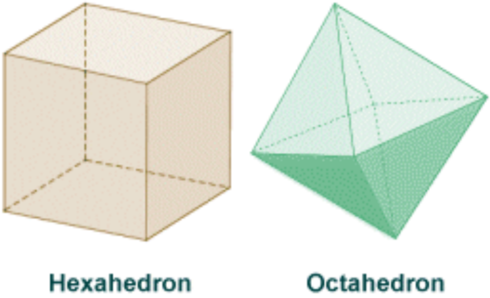


Tetrahedron

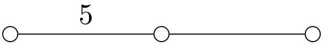
Platonic solids

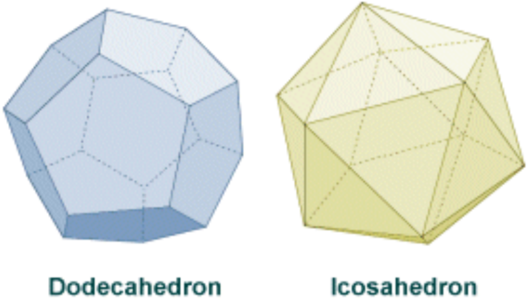
The group of type B_3 is the symmetric group of a cube/regular octahedron (八面体).





The group of type H_3 is the symmetric group of a regular dodecahedron/icosahedron (十二面体/二十面体).





Tits representations

Question: Given a Coxeter group (W, S) in general, can we realize W as a "reflection group" acting on some linear space?

- Let $V_{\text{Tits}} = \bigoplus_{s \in S} \mathbb{R}\alpha_s$. Define a bilinear form $(\cdot|\cdot)$ on V_{Tits} by

$$(\alpha_s|\alpha_t) = -\cos\frac{\pi}{m_{st}},$$

  here we regard $-\cos\frac{\pi}{\infty} = -1$, and $m_{ss} = 1$.
- In particular, $(\alpha_s|\alpha_s) = 1$, and $(\cdot|\cdot)$ is symmetric.
  (But NOT Euclidean in general!)
- $\forall s \in S$, define

$$\rho(s) : V_{\text{Tits}} \to V_{\text{Tits}}$$
$$x \mapsto x - 2(x|\alpha_s)\alpha_s$$

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Proposition

1. $\rho : W \to \mathrm{GL}(V_{\mathsf{Tits}})$ is a well defined representation of W.
2. $(wx|wy) = (x|y)$, $\forall w \in W$, $\forall x, y \in V_{\mathsf{Tits}}$.
3. ρ is faithful: $W \hookrightarrow \mathrm{GL}(V_{\mathsf{Tits}})$.
   (W is realized as a "reflection group" on V_{Tits}.)
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Reflection representations

Definition

1. A representation V of (W, S) is called a *reflection representation* if $\forall s \in S$, s acts by a "reflection", namely:
   - $\exists$ a subspace $H_s \subseteq V$ of $\operatorname{codim} 1$ s.t. $s|_{H_s} = \mathrm{Id}_{H_s}$;
   - $\exists \alpha_s \in V \setminus H_s$ s.t. $s(\alpha_s) = -\alpha_s$.
2. A reflection representation V is called an *R-representation* if $\{\alpha_s | s \in S\}$ spans V, and for any $s, t \in S$ such that $m_{st} < \infty$ it holds $\dim\langle \alpha_s, \alpha_t \rangle = 2$.
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For convenience, assume $m_{st} < \infty$, $\forall s, t \in S$.

Theorem (Classification of IR-representations, H. 2021)

1. The isomorphism classes of IR-representations one-to-one correspond to the following data:

$$\Big\{ \big((k_{st})_{s,t \in S, s \neq t}, \chi \big) \Big| 1 \leq k_{st} = k_{ts} \leq \frac{m_{st}}{2}, \forall s, t \in S, s \neq t; \\ \chi : \mathrm{H}_1(\widetilde{G}, \mathbb{Z}) \to \mathbb{R}^\times \text{ is a character} \Big\},$$

where $\widetilde{G}$ is a (undirected, unlabelled) graph with

- vertices set: S;
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Theorem (Continued)

2. There exists a nonzero W-invariant bilinear form $(\cdot|\cdot)$ on an IR-representation V if and only if $\operatorname{Im}\chi \subseteq \{\pm 1\}$.
   In this case, $(\cdot|\cdot)$ is symmetric and unique up to a scalar, and the action of $s \in S$ on V is the reflection w.r.t. α_s under $(\cdot|\cdot)$:

$$\sigma_{\alpha_s} : x \mapsto x - 2\frac{(x|\alpha_s)}{(\alpha_s|\alpha_s)}\alpha_s, \forall x \in V.$$

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Main idea: The two dimensional subspace $\langle \alpha_s, \alpha_t \rangle$ forms a subrepresentation $\rho_{k_{st}}$ of the dihedral group $\langle s, t \rangle$, determined by a number $1 \leq k_{st} \leq m_{st}/2$.

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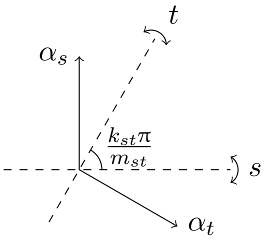

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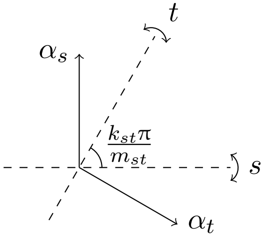

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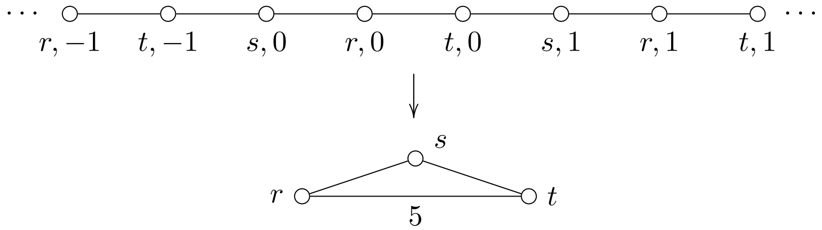

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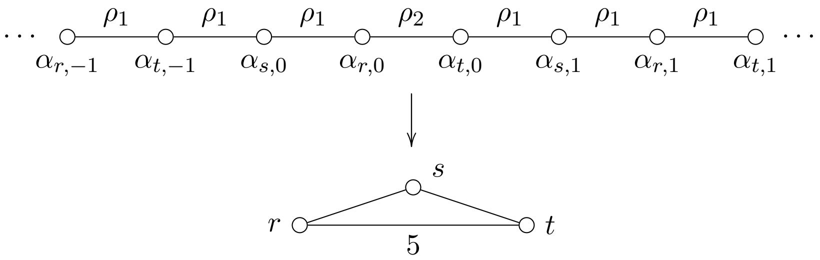

Let $V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}\langle \alpha_{s,n}, \alpha_{r,n}, \alpha_{t,n} \rangle$, define the action of s, t, r such that:

$\alpha_{r,0}, \alpha_{t,0}$ span a representation ρ_2 of $\langle r, t \rangle$;

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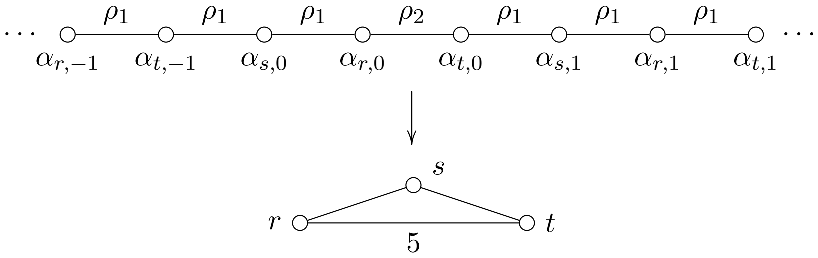

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Proposition (H. 2022)

If the Coxeter graph G of (W, S) satisfies one of the following, then W admits infinite dimensional irreducible representations (over $\mathbb{C}$).

1. G has at least two circuits.
2. G has at least one circuit, and $m_{st} \geq 4$ for some $s, t \in S$.

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What is further? (2. Lusztig's $\boldsymbol{a}$-function)

Kazhdan-Lusztig basis (1979): $\{C_w | w \in W\}$ a basis of $\mathbb{R}[W]$.

Lusztig's $\boldsymbol{a}$-function (1985): a function $\boldsymbol{a} : W \to \mathbb{N}$.

(Concrete definitions are omitted.)

It was conjectured that the function $\boldsymbol{a}$ is bounded on W if $|S| < \infty$.

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Let V be a reflection representation. The following are equivalent:

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I guess that the proportionality of those reflection vectors $\{\alpha_s\}_s$ is related to Lusztig's $\boldsymbol{a}$-function.

If so, the sign representation ($s \mapsto -1$, $\forall s \in S$, the smallest reflection representation) would be helpful to the boundedness conjecture.

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